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1980 J. Phys. A: Math. Gen. 13 1335

(http://iopscience.iop.org/0305-4470/13/4/025)

J. Phys. A: Math. Gen. **13** (1980) 1335–1354. Printed in Great Britain

The site percolation as a boundary problem

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Received 30 March 1979

Abstract. The site percolation on two-dimensional lattices is shown to be related to a boundary problem. This permits a new approach for the site percolation on a Cayley tree: it is shown that there is no percolation on a Cayley tree, if one takes care of the great number of boundary sites; this is consistent with the absence of spontaneous magnetisation on such lattices. However, the usually considered percolation on these lattices is shown to be consistent with the existence of a spontaneous local magnetisation for sites which are far from the boundary. Finally, the ferromagnetic percolation problem on usual lattices is also considered as a boundary problem.

1. Introduction

Until recently, the Bethe lattice was considered as the typical one for which exact results could be obtained, due to the fact that it contains no closed loops. For instance, the Bethe–Peierls approximation was shown to solve exactly the Ising problem (Domb 1960), or the site percolation problem was solved (Fisher and Essam 1961) with a critical concentration $x_c = (z-1)^{-1}$, where z is the coordination number.

However, there has been an increase of interest since Eggarter (1974) showed that the transition occuring on a Cayley tree is not of the type usually considered: a careful study was made by Matsuda (1974), showing that unusual properties are due to the non-vanishing proportion of boundary sites in the thermodynamic limit, and this has led to the discovery of transitions of continuous order (Müller–Hartmann and Zittartz 1974). We want to show here that such features are also present for the site percolation problem on a Cayley tree.

In § 2, we reformulate the site percolation problem (taking the square lattice as an example) in a manner showing its close connection with a boundary problem. Some remarks are also made on the dilute Ising problem.

In § 3, we show that, if one takes care of the great number of boundary sites for a Cayley tree, there is *no percolation* ($x_c = 1$) for it. As a consequence, there is no spontaneous magnetisation for the dilute Ising model on it.

In § 4, we define an unusual type of percolation problem, which is shown to coincide with the percolation problem considered by previous authors, leading to $x_c = (z-1)^{-1}$. In fact, the Cayley tree we consider in §§ 3 and 4 has $z = 3$ for simplicity, and we extend the results for $z > 3$ in § 5.

Returning to the usual lattices (square plane), the ferromagnetic percolation is considered in § 6 along the same lines. This needs a careful study to have the correct definitions, but one can see that here also it may be viewed as a boundary problem.

0305-4470/80/041335+20$01.50

2. The site percolation for a square lattice

Let us consider a square lattice of $N \times N$ sites. We consider the $2^{N \times N}$ configurations χ obtained by putting n atoms A on certain sites, and $N^2 - n$ atoms B on the remaining sites and attribute to each configuration where n atoms A are present the probability $x^n(1-x)^{N^2-n}$, where $0 \leqslant x \leqslant 1$ is given (Griffiths and Lebowitz 1968). Let us define a function f for each configuration: to each χ, we associate the number $f(\chi)$.

We shall now calculate

$$\bar{f} \equiv \sum_{\chi} f(\chi) x^n (1-x)^{N^2-n} \tag{1}$$

where the sum is over the 2^{N^2} configurations χ, and n is the number of atoms A in each of them.

Let us suppose that

$$f(\chi) \equiv \sum_{Cl(\chi)} \varphi(Cl) \tag{2}$$

where the sum is over all the clusters of A atoms which are present in the configurations χ; as usual, we call a cluster a set of sites which are connected by a path of nearest-neighbour bonds. Then one obtains:

$$\bar{f} = \sum_{Cl} \varphi(Cl) \sum_{\chi \supset Cl} x^n (1-x)^{N^2-n} \tag{3}$$

where the first sum is over all the clusters that are possibly found in any one of the 2^{N^2} configurations, and the second sum is over all the configurations containing a given cluster.

Now, one has:

$$\sum_{\chi \supset Cl} x^n (1-x)^{N^2-n} = x^p (1-x)^q \tag{4}$$

where p is the number of sites in the cluster, and q is the number of boundary sites, i.e. the number of sites of the lattice which are not in the cluster but are connected to some site of the cluster by a nearest-neighbour bond (figure 1).

Let us now suppose that $\varphi(Cl_1) = \varphi(Cl_2)$ if Cl_1 and Cl_2 are two clusters which can be superposed by a translation (Cl_1 and Cl_2 will be said to belong to the same type of cluster).

Then:

$$\bar{f} = \sum_{C} \varphi(C) \sum_{Cl \sim C} x^p (1-x)^q \tag{5}$$

where the first sum is over all the types of clusters that are possibly found in any one of the 2^{N^2} configurations, and the second sum is over all the clusters which are of the same type. Now, for each of them, p is the same but q is not the same: in fact, if the cluster is put on the lattice so that no site of it is on the boundary of the lattice, q is equal to the value it would have if the cluster was imbedded in the infinite lattice. If, however, the cluster is in a position such that some sites of it are on the boundary of the lattice, q is reduced from its standard value (figure 1).

We are thus led to characterise a type of cluster by the following eight parameters:

p: number of sites in the cluster

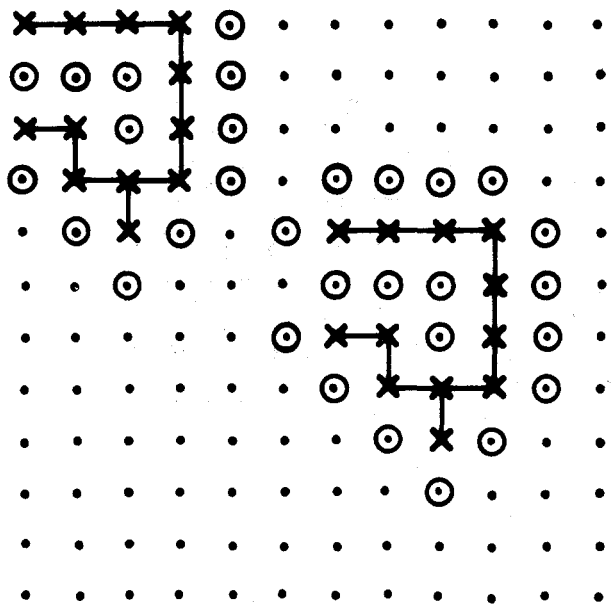

Figure 1. A configuration with two clusters of the same type; both have $p = 12$, but one has $q = 18$ and the other has $q = 12$. Sites occupied by A atoms are ×, sites occupied by B atoms are •, boundary sites for clusters are ⊙.

q: number of boundary sites if the cluster was imbedded in an infinite lattice

k, l: dimensions of the cluster

n_1, n_2, n_3, n_4: number of sites of the cluster that are the most to the left, the most to the top, the most to the right, the most to the bottom.

These six last parameters are determined in the following way: around each site of the cluster, we draw the four edges of the unit square centred at the site, then we suppress the edges that occur twice (Kunz and Souillard 1978). Then we consider the smallest rectangle which entirely contains the contour just obtained. We call k and l the lengths of the sides of the rectangle, and n_1, n_2, n_3, n_4 the length of the portions of the contour that coincide with some portion of the left, upper, right, lower edge of the rectangle respectively (figure 2).

Then it will be easily seen that:

$$\bar{f} = \sum_{\substack{C:k\leqslant N\\ l\leqslant N}} \varphi(C)[\rho_N(C)+\rho'_N(C)+\rho''_N(C)] \tag{6}$$

where the sum is over all types of clusters which can possibly occur, i.e. types with k and $l \leqslant N$:

$$\rho_N(C) = [1-\delta(k,N)][1-\delta(l,N)](N-1-l)(N-1-k)x^p(1-x)^q \tag{7}$$

$$\begin{aligned}\rho'_N(C) = {} & [1-\delta(k,N)][1-\delta(l,N)](N-1-l)x^p[(1-x)^{q-n_2}+(1-x)^{q-n_4}]\\ & +[1-\delta(k,N)][1-\delta(l,N)](N-1-k)x^p[(1-x)^{q-n_1}+(1-x)^{q-n_3}]\\ & +[1-\delta(k,N)][1-\delta(l,N)]x^p[(1-x)^{q-n_1-n_2}\\ & +(1-x)^{q-n_2-n_3}+(1-x)^{q-n_3-n_4}+(1-x)^{q-n_4-n_1}]\end{aligned} \tag{8}$$

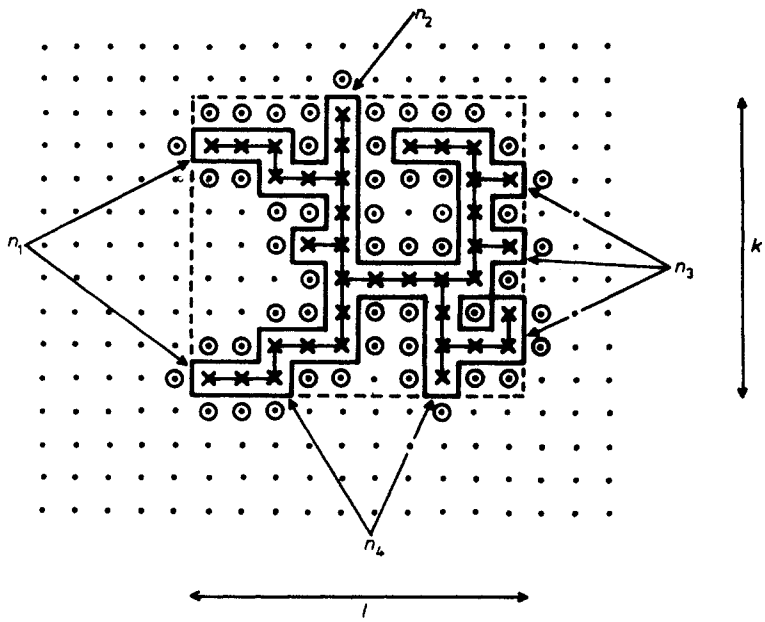

Figure 2. The parameters defining a cluster. Here $p = 37$, $q = 52$, $k = 9$, $l = 10$, $n_1 = 2$, $n_2 = 1$, $n_3 = 4$, $n_4 = 4$.

$$\begin{aligned}\rho''_N(C) = {} & [1-\delta(l,N)]\delta(k,N)x^p[(N-1-l)(1-x)^{q-n_2-n_4} \\ & +(1-x)^{q-n_2-n_4-n_1}+(1-x)^{q-n_2-n_4-n_3}] \\ & +[1-\delta(k,N)]\delta(l,N)x^p[(N-1-k)(1-x)^{q-n_1-n_3}+(1-x)^{q-n_1-n_3-n_2} \\ & +(1-x)^{q-n_1-n_3-n_4}]+\delta(k,N)\delta(l,N)x^p(1-x)^{q-n_1-n_2-n_3-n_4}. \end{aligned} \tag{9}$$

Here, $\rho_N(C)$ corresponds to clusters in positions such that no site is on the boundary of the $N \times N$ lattice, contrary to $\rho'_N(C)$; $\rho''_N(C)$ corresponds to clusters that are so large that, necessarily, they have sites on the boundary of the lattice.

As an example, we shall take the function f which associates to each configuration the number of A atoms in it; such a function can be defined as

$$f(\chi) = \sum_{Cl} \varphi(Cl) \tag{10}$$

with $\varphi(Cl) \equiv p$, i.e. the number of A atoms in the cluster.

On the other hand, one can calculate directly

$$\bar{f} = \sum_{\chi} nx^n(1-x)^{N^2-n} = N^2x \tag{11}$$

so that we obtain the main identity:

$$N^2x \equiv \sum_{\substack{C:k\leq N \\ l\leq N}} p[\rho_N(C)+\rho'_N(C)+\rho''_N(C)]. \tag{12}$$

It must be emphasised that, up to now, we have worked on finite lattices and all equalities result from elementary considerations.

Now, we shall rewrite (12) in the form

$$x \equiv \sum_{\substack{C:k\leqslant N\\ l\leqslant N}} \frac{p}{N^2}[\rho_N(C)+\rho'_N(C)+\rho''_N(C)]. \tag{13}$$

Let $N\to\infty$. A simple guess would be the following: ρ'_N and ρ''_N being associated with boundary effects, they probably cancel in the limit, so that

$$x = \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N\\ l\leqslant N}} \frac{p}{N^2}\rho_N(C). \tag{14}$$

We shall now see that (14) is wrong because ρ'_N and ρ''_N are precisely responsible for the occurrence of percolation.

Let $(\bigcup_1^s C_i)$ be a given family of types of clusters, and let $g(\bigcup_1^s C_i)$ be the mean number of A atoms which are in this family, i.e. the number of such atoms divided by the number of sites N^2, when $N\to\infty$. Then N is larger than the dimensions k and l of any of the C_i, so that

$$g\left(\bigcup_1^s C_i\right) = \lim_{N\to\infty} \sum_{(C\in\bigcup_1^s C_i)} \frac{p}{N^2}[\rho_N(C)+\rho'_N(C)] \tag{15}$$

and, because s is fixed and the k, l, p, q, n_1, n_2, n_3, n_4 of each C_i are given, one obtains

$$g\left(\bigcup_1^s C_i\right) = \sum_{(C\in\bigcup_1^s C_i)} px^p(1-x)^q. \tag{16}$$

Letting $s\to\infty$, i.e. taking all the possible families of types of clusters, we obtain the function

$$g(x) = \sum_{(C)} px^p(1-x)^q \tag{17}$$

which will be recognised as the usual percolation series.

This is indeed an infinite series and we shall now see that this series is convergent, i.e. that expression (17) makes sense for every x.

In fact, let us consider the mean number of A atoms which are not connected by a path of sites A to some site (included) of the boundary of the $N\times N$ lattice; let $\varphi(x)$ be this number, in the limit $N\to\infty$. One then has:

$$\varphi(x) = \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N-1\\ l\leqslant N-1}} \frac{p}{N^2}\rho_N(C). \tag{18}$$

Let M be fixed: one can now write

$$x \geqslant \varphi(x) \geqslant \lim_{N\to\infty} \sum_{\substack{C:k\leqslant M\\ l\leqslant M}} (N-1-M)^2 \frac{p}{N^2} x^p(1-q)^q$$

$$= \sum_{\substack{C:k\leqslant M\\ l\leqslant M}} px^p(1-x)^q \equiv S_M. \tag{19}$$

The sequence S_M is clearly increasing in M, bounded by x, and so it has a limit when $M \to \infty$.

This limit is clearly $g(x)$ because, due to the positiveness of terms in (17), we can rearrange them in any order we want.

Thus

$$x \geqslant \varphi(x) \geqslant g(x). \tag{20}$$

On the other hand:

$$\varphi(x) \leqslant \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N \\ l\leqslant N}} (N-2)^2 \frac{p}{N^2} x^p (1-x)^q$$

$$= \lim_{N\to\infty} \left(\frac{N-2}{N}\right)^2 S_N = g(x) \tag{21}$$

so that, using (20) and (21), we obtain:

$$g(x) = \varphi(x) \leqslant x. \tag{22}$$

Thus, in this case, the sum of the percolation series, which represents the mean number of atoms A which are in any family of (finite) clusters, is equal to the mean number of atoms A which are in clusters not connected to the boundary of the $N \times N$ lattice, when $N \to \infty$.

The so-called infinite cluster may be viewed as the set of atoms A which are in clusters connected to the boundary of the $N \times N$ lattice. This has been used recently to obtain rigorous results for the percolation problem (Kunz and Souillard 1978).

Thus we have:

$$x - g(x) = \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N \\ l\leqslant N}} \frac{p}{N^2} [\rho'_N(C) + \rho''_N(C)]. \tag{23}$$

Based on numerical calculations (Essam 1972, Sykes *et al* 1976):

$$x - g(x) \equiv 0 \qquad \text{for} \qquad x \leqslant x_c \sim 0{\cdot}59. \tag{24}$$

It is shown in appendix 1 how these considerations lead very easily to $g(x) = x$ for low x.

Let us now briefly discuss the quenched dilute Ising model (Griffiths and Lebowitz 1968). We imagine that atoms A carry a localised spin $S = \frac{1}{2}$, and that spins interact via the usual nearest-neighbour ferromagnetic Ising interaction; moreover, they are put in a magnetic field h. For each configuration, we can calculate the mean magnetisation per site (using the Boltzmann distribution). Then, we take the average over the 2^{N^2} configurations, let $N \to \infty$, and finally let $h \to +0$. It must be emphasised that the order of the preceding operations is of fundamental importance.

However it is clear that, if we denote by m_c the magnetisation per atom in a given type of cluster, we have:

$$N^2 m_N \equiv \sum_{\substack{C:k\leqslant N \\ l\leqslant N}} p m_c [\rho_N(C) + \rho'_N(C) + \rho''_N(C)] \tag{25}$$

where $N^2 m_N$ represents the global magnetisation averaged over the 2^{N^2} configurations.

Then, if m is the mean magnetisation per site, one obtains:

$$m = \sum_{(C)} p m_c x^p (1-x)^q + \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N\\ l\leqslant N}} \frac{p}{N^2} m_c [\rho'_N(C) + \rho''_N(C)]. \tag{26}$$

The first term on the right-hand side is a convergent series, bounded (because $0 \leqslant m_c \leqslant 1$ in case of $h \geqslant 0$ and ferromagnetic interactions) by the percolation series. It is thus a continuous function of $h \geqslant 0$, because of uniform convergence and, as $m_c(h) \to 0$ when $h \to 0$, one obtains:

$$\lim_{h\to+0} \sum_{(C)} p m_c x^p (1-x)^q = 0 \tag{27}$$

so that, for every x:

$$\lim_{h\to+0} m = \lim_{h\to+0} \lim_{N\to\infty} \sum_{\substack{C:k\leqslant N\\ l\leqslant N}} \frac{P}{N^2} m_c [\rho'_N(C) + \rho''_N(C)]. \tag{28}$$

Moreover, using Griffiths' inequalities (Griffiths 1967, p 478) and equality between local and thermodynamic magnetisation for the homogeneous (non-random) Ising model (Griffiths 1967, p 484), it will be shown that m_c is less than the latter. Letting then $h \to +0$, which defines the usual spontaneous magnetisation m_s per site for the homogeneous Ising model, one obtains:

$$\lim_{h\to+0} m \leqslant [x - g(x)] m_s \tag{29}$$

so that we have certainly no spontaneous magnetisation for the quenched Ising model if $x \leqslant x_c$, or if $x > x_c$ for temperatures higher than the usual Curie temperature for the homogeneous Ising model. We shall now apply these ideas to the Cayley tree and, first, return to the percolation problem itself.

3. The site percolation for a Cayley tree ($z = 3$)

Let us consider a Cayley tree with coordination number $z = 3$ and of order N: we consider a central site, numbered 1. Then, we construct, upwards, the two neighbours (layer 2), then, upwards, the two neighbours of the preceding ones, etc, up to the layer N (figure 3). We note that the central point has two neighbours, the points on the layer N (boundary) have one neighbour, and all the other points have $z = 3$ neighbours.

We imagine that a cluster is made of rigid branches (pairs inside the cluster) free to move around each point of the cluster considered as an articulation point: then, two clusters will be said to be of the same type if they can be superposed by a translation (figure 3).

We shall then characterise a type of cluster by the five following parameters:

p: number of sites in the cluster

q: number of boundary sites if the cluster was imbedded in an infinite lattice (with no site coinciding with the centre of the lattice) $q = p + 2$

k: height of the cluster

n: number of superficial points

t: number of oscillating points.

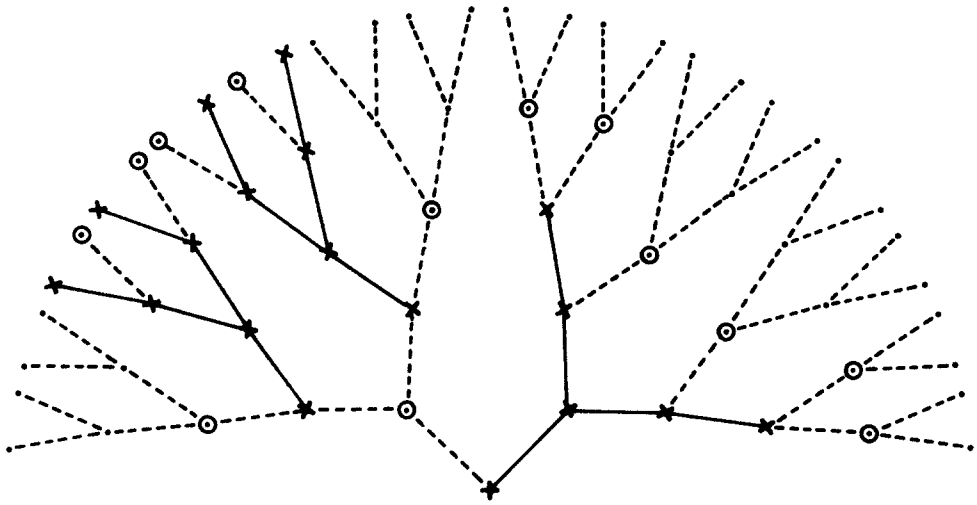

Figure 3. A configuration with three clusters of the same type, having $p = 6$, but $q = 4$, 4 and 7, respectively.

These three last parameters are determined in the following way: if we imagine that the type of cluster is put with its centre (the site lying on the lowest layer) coinciding with the centre of the lattice, it would have sites on layers 1, 2, . . . , up to k; n is the number of points on the layer k. Finally, we say that a point is an oscillating point if it has one pair upwards (figure 4).

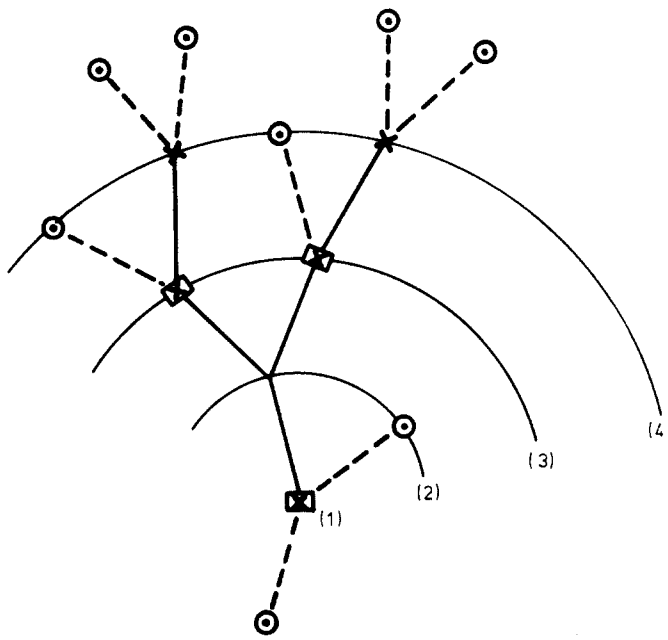

Figure 4. The parameters defining a cluster. Here $p = 6$, $q = 8$, $k = 4$, $n = 2$, $t = 3$. The oscillating points are ☒.

To put in place a cluster of a given type, we put in place its centre, successively on the different points of the tree, and then make it 'oscillate' around its oscillating points because, if we have one pair upwards, it may occupy two different positions corresponding to the two possible neighbours that are upwards.

The total number of sites being 2^N-1, it will be easily seen that one has

$$(2^N-1)x \equiv \sum_{\{C:k\leqslant N\}} p\,[\rho_N(C)+\rho'_N(c)+\rho''_N(C)] \tag{30}$$

with

$$\rho_N(C)=[1-\delta(k,N)]2^t(2^{N-k}-2)x^p(1-x)^{p+2} \tag{31}$$

$$\rho'_N(C)=[1-\delta(k,N)]2^t[2^{N-k}x^p(1-x)^{p+2-2n}+x^p(1-x)^{p+1}] \tag{32}$$

$$\rho''_N(C)=\delta(k,N)2^t x^p(1-x)^{p+1-2n} \tag{33}$$

corresponding to the similar expressions (12) and (7)–(9).

Let us now calculate the mean number of A atoms which are in a given family of types of cluster $(\bigcup_1^s C_i)$:

$$g\Big(\bigcup_1^s C_i\Big)=\lim_{N\to\infty}\sum_{(C\in\bigcup_1^s C_i)}\frac{p}{2^N-1}[\rho_N(C)+\rho'_N(C)] \tag{34}$$

i.e.

$$g\Big(\bigcup_1^s C_i\Big)=\sum_{(C\in\bigcup_1^s C_i)} px^p(1-x)^{p+2}\frac{2^t}{2^k}[1+(1-x)^{-2n}] \tag{35}$$

so that the percolation series is now

$$g(x)=\sum_{(C)} px^p(1-x)^{p+2}\frac{2^t}{2^k}[1+(1-x)^{-2n}]. \tag{36}$$

It must be emphasised that, contrary to the case of the square lattice, the term ρ'_N corresponding to clusters having sites on the boundary of the Cayley tree has given a contribution to the percolation series; in other words, the series $g(x)$ is not the series usually considered (Fisher and Essam 1961) when one 'neglects' boundary effects.

To be still more concrete, we give the first terms obtained by ordering the (positive) series in ascending powers of x^p:

$$\begin{aligned} g(x)=(x/2)[(1-x)^3+(1-x)]+x^2[(1-x)^4+(1-x)^2] \\ +(3x^3/4)[3(1-x)^5+2(1-x)^3+(1-x)] \\ +x^4[5(1-x)^6+4(1-x)^4+(1-x)^2]+\ldots. \end{aligned} \tag{37}$$

It will be shown that, as usual for percolation series (Sykes and Glen 1976), every power of x (but the first) vanishes in this low-density expansion, i.e., if we develop the terms written above, we obtain:

$$g(x)=x+\text{terms beginning with } x^5.$$

Now, we come to the main result: the series $g(x)$ is convergent for every $x(0 \leqslant x \leqslant 1)$ and

$$\text{for } 0 \leqslant x < 1 \qquad g(x) = x. \tag{38}$$

(It is clear that $g(1) = 0$.)

This is rather easy to obtain, but necessitates some calculations that we have reproduced in appendix 2.

This means that, if we take into account the large number of boundary sites in the Cayley tree, there is *no percolation* in the whole range $0 \leqslant x < 1$. This is the exact analogue of the result for the linear chain (Cayley tree with $z = 2$).

If we come now to the quenched Ising model on the Cayley tree, defining m_c as the magnetisation per atom in a given cluster and m as the magnetisation per site in the quenched system, we obtain for $0 \leqslant x < 1$

$$m = \sum_{(C)} p x^p (1-x)^{p+2} m_c \frac{2^t}{2^k} [1 + (1-x)^{-2n}] \tag{39}$$

because we have no percolation. This gives the result that:

$$\lim_{h \to +0} m = 0 \tag{40}$$

for $0 \leqslant x < 1$. On the other hand, it has been demonstrated (Eggarter 1974, Matsuda 1974) that, in the pure system ($x = 1$), one also has no spontaneous magnetisation, i.e. equation (40) is now valid for every $0 \leqslant x \leqslant 1$. In the pure system, the result clearly comes (Matsuda 1974) from the large number of boundary sites relative to the total number of sites: we see that the same phenomenon occurs in the random case.

We shall now come to a particular type of percolation which reproduces the results usually quoted for Bethe lattices ($x_c = \frac{1}{2}$).

4. The percolation near the centre of a Cayley tree ($z = 3$)

We have just seen that, if $\langle S_i \rangle$ ($i = 1, \ldots, 2^N - 1$) means the thermal average of the spin at site i (when present), one has:

$$\lim_{h \to +0} \lim_{N \to \infty} \overline{\left(\frac{1}{2^N - 1} \sum_i \langle S_i \rangle \right)} = 0 \tag{41}$$

where the upper bar means a configurational average. Now, it has been shown (Matsuda 1974) that, in the pure case, local magnetisations near the centre of the tree have the usual features of spontaneous magnetisation in phase transitions problems. Considering, for instance, the magnetisation at the centre of the tree, one has:

$$\lim_{h \to +0} \lim_{N \to \infty} \langle S_1 \rangle \neq 0 \tag{42}$$

if the temperature T is lower than T_{BP}, the temperature calculated by the so-called Bethe–Peierls approximation.

Now, in the random case ($x \neq 1$), let us calculate the magnetisation at the centre of the tree.

For a finite tree, we have

$$\langle S_1\rangle = \langle S_1\rangle_C \tag{43}$$

for clusters C containing site (1), so that

$$\overline{\langle S_1\rangle} = \sum_{\{C:k\leqslant N\}} \langle S_1\rangle_C \{[1-\delta(k,N)]2^t x^p(1-x)^{p+1} + \delta(k,N)2^t x^p(1-x)^{p+1-2n}\}. \tag{44}$$

In the corresponding percolation problem, we have to calculate the number of clusters which contain site (1), i.e. we sum the probability law of configurations over all configurations having site (1) occupied by an A atom and we thus obtain:

$$x = \sum_{\{C:k\leqslant N\}} \{[1-\delta(k,N)]2^t x^p(1-x)^{p+1} + \delta(k,N)2^t x^p(1-x)^{p+1-2n}\} \tag{45}$$

(which can be easily checked for N not too large, if desired).

We emphasise the differences between (45) and (30): here, we count the number of clusters and have no factor p in the right-hand side. We restrict ourselves to clusters containing the centre of the tree, and we have simply x in the left-hand side.

Now, if we calculate the number of clusters which contain site (1) and which belong to a given family of types of clusters $\bigcup_1^s C_i$, then, if we let $s\to\infty$, we obtain a new percolation series:

$$\psi(x) = \sum_{(C)} x^p(1-x)^{p+1}2^t \tag{46}$$

(to be compared with $g(x)$ given by (36)).

The first terms are now:

$$\psi(x) = x(1-x)^2 + 2x^2(1-x)^3 + 5x^3(1-x)^4 + 14x^4(1-x)^5 + \ldots. \tag{47}$$

Now (46) may be written:

$$\psi(x) = (1-x)\sum_{p=1}^{\infty} [x(1-x)]^p g_p \tag{48}$$

where

$$g_p = \sum_{\{C:p\}} 2^t \tag{49}$$

is the number of clusters of p sites containing site (1) in an infinite Cayley tree. One can easily obtain a recurrence formula for the g_p, leading to

$$g_p = \frac{(2p)!}{p!(p+1)!} \tag{50}$$

so that (48) can be summed, giving:

$$\begin{aligned} &\psi(x) = x && \text{if } x \leqslant \tfrac{1}{2} \\ &\psi(x) = (1-x)^2/x && \text{if } x > \tfrac{1}{2}. \end{aligned} \tag{51}$$

We thus have percolation for $x > \frac{1}{2}$, and this is the result usually quoted for Cayley trees. Here, we have explicitly shown the nature of the approximation involved when 'neglecting boundary effects'.

As a consequence for the dilute Ising model, one clearly has

$$\lim_{h\to+0}\lim_{N\to\infty}\overline{\langle S_1\rangle}=0 \qquad \text{for } x\leqslant\tfrac{1}{2} \tag{52}$$

and it remains for us to study

$$\overline{\langle S_1\rangle}=\lim_{h\to+0}\lim_{N\to\infty}\sum_{\{C:k=N\}}\langle S_1\rangle_C 2^t x^p(1-x)^{p+1-2n} \tag{53}$$

which can be possibly non-zero only for $x>\frac{1}{2}$. In particular, it would be interesting to know if $\lim_{T\to0}\overline{\langle S_1\rangle}$ (note the order of taking limits: $N\to\infty$, then $h\to+0$, then $T\to0$) is non-zero as soon as $x>\frac{1}{2}$, but we do not know any rigorous way to show it.

Finally, by analogy with the pure case, we expect that similar results occur for clusters containing a given site remaining at finite distance from the centre when the boundary goes to infinity.

5. Percolation for a Cayley tree with any coordination number

The generalisation of the above results for $z>3$ is obvious. The number of boundary sites (in an infinite lattice) will be:

$$q=p(z-2)+2.$$

We must now define more general oscillating points; we say that a point is an oscillating point of order $m(=1,2,\ldots,z-2)$ if it has m branches leaving it upwards.

Now, if α_m is the number of oscillating points of order m, (30)–(33) will be replaced by

$$\frac{(z-1)^N-1}{z-2}x=\sum_{\{C:k\leqslant N\}}p[\rho_N(C)+\rho'_N(C)+\rho''_N(C)] \tag{54}$$

with

$$\rho_N(C)=[1-\delta(k,N)]2^t\frac{(z-1)^{N-k}-(z-1)}{z-2}x^p(1-x)^{(z-2)p+2} \tag{55}$$

$$\rho'_N(C)=[1-\delta(k,N)]2^t[(z-1)^{N-k}x^p(1-x)^{(z-2)p+2-n(z-1)}+x^p(1-x)^{(z-2)p+1}] \tag{56}$$

$$\rho''_N(C)=\delta(k,N)2^t x^p(1-x)^{(z-2)p+1-n(z-1)} \tag{57}$$

where 2^t now means:

$$2^t\equiv(C^1_{z-1})^{\alpha_1}(C^2_{z-1})^{\alpha_2}\ldots(C^{z-2}_{z-1})^{\alpha_{z-2}} \tag{58}$$

with

$$C^m_{z-1}=\frac{(z-1)!}{m!(z-1-m)!}.$$

The percolation series will be

$$g(x)=\sum_{(C)}p\,x^p(1-x)^{p(z-2)+2}\frac{2^t}{(z-1)^k}[1+(z-2)(1-x)^{-n(z-1)}]. \tag{59}$$

By straightforward extension of the calculations of appendix 2, it will be easily shown that, here again, there is no percolation for $x \neq 1$:

$$0 \leqslant x < 1 \qquad g(x) = x \tag{60}$$

and this implies no spontaneous magnetisation for the quenched dilute Ising model.

If we come now to the percolation series giving the number of clusters containing the centre of the tree, we shall have:

$$\psi(x) = (1-x) \sum_{p=1}^{\infty} [x(1-x)^{z-2}]^p g_p \tag{61}$$

with

$$g_p = \frac{[(z-1)p]!}{p![(z-2)p+1]!}. \tag{62}$$

It will be shown that $\psi(x)$ is the smallest root of

$$x(1-x+\psi)^{z-1} = \psi. \tag{63}$$

In particular, we obtain $\psi = x$, i.e. no percolation, for $x \leqslant (z-1)^{-1}$. This is merely the standard result known for Cayley trees (Fisher and Essam 1961, Coniglio 1976).

6. Ferromagnetic percolation

We shall now return to the case of usual lattices. Although several interesting features are known from various approaches, rigorous results are still lacking for the percolation problem (Wierman 1978). Needless to say, the situation is still more complex for the correlated percolation problem. We now want to show that our approach (§ 2) is also valid for this type of problem.

Let us then consider the infinite square lattice, with some origin O, and the squares $N \times N$ centred on O.

We consider now

$$H_N = -J \sum_{(ij)} S_i S_j - h \sum_i S_i \tag{64}$$

with $J, h > 0$; (ij) stands for the set of pairs of nearest-neighbour bonds between sites i and j belonging to the square; the set $\{S_i = \pm 1\}$ defines $2^{N \times N}$ configurations, and we associate to each of them the probability

$$\rho(\chi) = \exp(-\beta H_N)/\mathrm{Tr} \exp(-\beta H_N) \tag{65}$$

where Tr means a sum over the values $S_i = \pm 1$ for $i = 1$ to N^2. We note that $\rho(\chi) > 0$ and $\Sigma_\chi \rho(\chi) = \mathrm{Tr}\, \rho(\chi) = 1$. In other words, the configurational average defined in § 2 is now replaced by the usual thermal average, which we shall indicate by brackets:

$$\langle \ldots \rangle_N = \mathrm{Tr} \ldots \exp(-\beta H_N)/\mathrm{Tr} \exp(-\beta H_N). \tag{66}$$

Suppose now that we want to calculate the average number of up spins:

$$\sum_\chi \left(\sum_i \frac{1+S_i}{2} \right) \rho(\chi) = \mathrm{Tr} \left(\sum_i \frac{1+S_i}{2} \right) \rho(\chi) = \left\langle \sum_i \frac{1+S_i}{2} \right\rangle_N. \tag{67}$$

We can decompose it in a sum over all possible clusters of up spins, so that, by the same arguments as in § 2:

$$\left\langle \sum_i \frac{1+S_i}{2} \right\rangle_N = \sum_{Cl} p \sum_{\chi \supset Cl} \rho(\chi) \tag{68}$$

where p is the number of spins (up) in the cluster considered.

Now, it is easy to see that:

$$\sum_{\chi \supset Cl} \rho(\chi) = \left\langle \prod_{i \in Cl} \frac{1+S_i}{2} \prod_{i \in \overline{Cl}} \frac{1-S_i}{2} \right\rangle_N \tag{69}$$

where $\overline{Cl}$ stands for the set of sites which are boundary sites for the cluster. In the case where $J = 0$, one recovers the results of § 2, with x defined by

$$x = \frac{\exp(\beta h)}{\exp(\beta h) + \exp(-\beta h)}. \tag{70}$$

The most striking difference with the results of § 2 is now that each term like (69) depends explicitly on the position of the cluster and N, and this makes the analysis much more difficult. We shall however follow the different steps of § 2, and see how they are modified here.

Let $(\bigcup_1^s C_r)$ be a given family of types of clusters, and let $g\,(\bigcup_1^s C_r)$ be the average number (divided by N^2) of up spins which are in this family, for $N \to \infty$. We then have:

$$g\left(\bigcup_1^s C_r\right) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{(Cl \in \bigcup_1^s C_r)} p \left\langle \prod_{i \in Cl} \frac{1+S_i}{2} \prod_{i \in \overline{Cl}} \frac{1-S_i}{2} \right\rangle_N. \tag{71}$$

Because s is given we can write:

$$g\left(\bigcup_1^s C_r\right) = \sum_{r=1}^{s} A_r \tag{72}$$

with

$$A_r = \lim_{N\to\infty} \frac{1}{N^2} \sum_{Cl \in C_r} p \left\langle \prod_{i \in Cl} \frac{1+S_i}{2} \prod_{i \in \overline{Cl}} \frac{1-S_i}{2} \right\rangle_N. \tag{73}$$

Let Cl be a given cluster with p sites and q boundary sites in the family C_r:

$$\left\langle \prod_{i \in Cl} \frac{1+S_i}{2} \prod_{i \in \overline{Cl}} \frac{1-S_i}{2} \right\rangle_N = \left\langle \frac{1}{2^{p+q}} + \frac{S_1}{2^{p+q}} + \dots \right\rangle_N \tag{74}$$

which is a sum of 2^{p+q} (finite) terms of type $\langle S_i \dots S_n \rangle_N$, with a coefficient ± 1.

If we make the same (finite) development for each cluster of C_r, we obtain:

$$g\left(\bigcup_1^s C_r\right) = \sum_{r=1}^{s} \sum \lim_{N\to\infty} \frac{1}{N^2} (-1)^{\pm 1} \sum_{Cl \in C_r}^{s} p \langle S_i \dots S_n \rangle_N \tag{75}$$

where the second sum is over 2^{p+q} (depending on r) terms.

If $i, \dots, n$ is a fixed set of sites, it is known (Griffiths 1967, p 484) that:

$$\lim_{N\to\infty} \langle S_i \dots S_n \rangle_N = \langle S_i \dots S_n \rangle \tag{76}$$

exists and that the limit is invariant in a translation of a lattice vector.

Here, we are not in this situation, because the third sum in (75) is over $(N-1-l)(N-1-k)$ terms (see § 2). However, by a straightforward extension of Griffiths' arguments on equivalence between local magnetisation and mean (thermodynamic) magnetisation, it is shown in appendix 3 that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{Cl\in C_r} \langle S_i \dots S_n \rangle_N = \langle S_i \dots S_n \rangle. \tag{77}$$

This in turn implies that

$$g\left(\bigcup_1^s C_r\right) = \sum_{r=1}^{s} p_r \left\langle \prod_{i\in C_r} \frac{1+S_i}{2} \prod_{i\in \bar{C}_r} \frac{1-S_i}{2} \right\rangle \tag{78}$$

where the bracket is the limit of an analogous term for a fixed cluster of the family r.

Summing over all types of clusters, i.e. taking the sup of (78), this gives the percolation series

$$g = \sum_{(C)} p \left\langle \prod_{i\in C} \frac{1+S_i}{2} \prod_{i\in \bar{C}} \frac{1-S_i}{2} \right\rangle. \tag{79}$$

This series is convergent because, as in § 2, if we denote by $\langle N^{\uparrow}/N^2 \rangle_{\text{int}}$ the average number of spins up which are in clusters not connected to the boundary of the square $N \times N$, we shall have, for every fixed M:

$$\left\langle \frac{1+S_i}{2} \right\rangle \geqslant \lim \left\langle \frac{N^{\uparrow}}{N^2} \right\rangle_{\text{int}} \geqslant S_M \tag{80}$$

where

$$S_M = \sum_{C_{(M)}} p \left\langle \prod_{i\in C} \frac{1+S_i}{2} \prod_{i\in \bar{C}} \frac{1-S_i}{2} \right\rangle \tag{81}$$

is restricted to families which can appear in a square $M \times M$. The sequence S_M is increasing, bounded, and has thus a limit which is clearly g.

We shall now stop here, with:

$$\frac{1+m}{2} \geqslant \lim \left(\frac{N^{\uparrow}}{N^2} \right)_{\text{int}} \geqslant g \tag{82}$$

with $m = \langle S_i \rangle$ because, contrary to the case of § 2, we are not able, at this point, to show that

$$g \geqslant \lim (N^{\uparrow}/N^2)_{\text{int}}. \tag{83}$$

However, it is clear that all that has been said is equally valid for clusters of spins down, so that

$$\frac{1-m}{2} \geqslant \lim \left(\frac{N^{\downarrow}}{N^2} \right)_{\text{int}} \geqslant g^{\downarrow} \tag{84}$$

with

$$g^{\downarrow} = \sum_{(C)} p \left\langle \prod_{i\in C} \frac{1-S_i}{2} \prod_{i\in \bar{C}} \frac{1+S_i}{2} \right\rangle \tag{85}$$

but here, we can show that

$$g^{\downarrow} > \lim (N^{\downarrow}/N^2)_{\text{int}}. \tag{86}$$

In fact, for a given cluster Cl, if we put:

$$\rho_{Cl} = \prod_{i \in Cl} \frac{1-Si}{2} \prod_{i \in \overline{Cl}} \frac{1+S_i}{2} \tag{87}$$

$$\langle \ldots \rangle_{Cl} = \mathrm{Tr} \ldots \exp(-\beta H_{Cl}) / \mathrm{Tr} \exp(-\beta H_{Cl}) \tag{88}$$

where H_{Cl} is the restriction of H_N to terms with $i \in Cl$, or (ij) with i or $j \in (Cl)$, it will be easily seen that:

$$\langle \rho_{Cl} \rangle_N = \frac{\langle \rho_{Cl} \rangle_{Cl}}{\langle \prod_{i \in \overline{Cl}} (1+S_i)/2 \rangle_{Cl}} \left\langle \prod_{i \in \overline{Cl}} \frac{1+S_i}{2} \right\rangle_N \tag{89}$$

(see Lebowitz and Penrose (1977) for an analogous result).

Now, two clusters belonging to the same family differ only via the term $\langle \prod_{i \in \overline{Cl}} (1+S_i)/2 \rangle_N$ and Griffiths' inequalities can be applied to this term (sum of correlation functions with all positive coefficients):

$$\left\langle \prod_{i \in \overline{Cl}} \frac{1+S_i}{2} \right\rangle_N \leqslant \lim_{N \to \infty} \left\langle \prod_{i \in \overline{Cl}} \frac{1+S_i}{2} \right\rangle_N = \left\langle \prod_{i \in \overline{Cl}} \frac{1+S_i}{2} \right\rangle. \tag{90}$$

This can be done for the $(N-1-l)(N-1-k)$ terms of a particular family, so that

$$\left(\frac{N^{\downarrow}}{N^2}\right)_{\mathrm{int}} \leqslant \frac{(N-2)^2}{N^2} S_N(\downarrow) \tag{91}$$

where $S_N(\downarrow)$ is the analogue of S_M given by (81), but now for spins down. Thus, we have shown that:

$$\lim (N^{\downarrow}/N^2)_{\mathrm{int}} = g^{\downarrow} \tag{92}$$

so that the infinite cluster of spins down, corresponding to the average number $[(1-m)/2] - g^{\downarrow}$ can be viewed as the set of spins down which are connected to the boundary of the square when this boundary goes to infinity.

Although we have no true argument, we think that this may have some connection with the problem of the phase transitions viewed as a macroscopic instability with respect to boundary conditions (Gallavotti (1972) and references therein).

Note that, if $h = 0$ (from the beginning), then $m = 0$, $g^{\downarrow} = g^{\uparrow} = \lim (N^{\uparrow}/N^2)_{\mathrm{int}}$. On the other hand, as previously said, we have not been able to show that, when $h \neq 0$, $g^{\uparrow} = \lim (N^{\uparrow}/N^2)_{\mathrm{int}}$. Moreover, if not true, we do not know if this may have consequences for the percolation problem of spins up.

A similar study would be possible for Cayley trees, and a theory 'neglecting surface effects' has appeared (Coniglio 1975).

As a final comment, we should wish to point out that we have not studied the problem of $\lim_{h \to +0} g$, but we hope that the approach that we have used here can throw some light on questions which may arise.

7. Conclusions

Instead of starting with the infinite lattice, as is usually done in percolation problems, we have carefully examined the finite lattice problem; the usefulness of such an approach

had been shown (Griffiths and Lebowitz 1968) for obtaining rigorous results in the dilute Ising problem.

Taking as a definition for the percolation series the mean number of sites in any (finite) cluster, we have shown that:

(a) In usual lattices, due to the vanishing ratio of number of surface sites over volume sites, this is equivalent to computing the mean number of sites in clusters which are not connected to the boundary.

(b) In Cayley trees, this is not true and results in the absence of percolation (for $x \neq 1$), and consequently in the absence of spontaneous magnetisation for the dilute Ising model. However, it is possible to define some particular type of percolation, 'near the centre of the tree', and one recovers then the critical concentration $x_c = (z-1)^{-1}$.

(c) It is possible to generalise the results for the ferromagnetic percolation problem; in a pure Ising model, it is possible to define percolation series for the average number of spins up (or down); in usual lattices, this corresponds, for spins down, to the average number of spins down which belong to clusters not connected to the boundary.

Appendix 1.

Although it is known that $x_c \sim 0{\cdot}593$ for the square lattice, we think it valuable to have rather direct proofs that $g(x) \equiv x$ for low x.

We reproduce equation (23) of the text:

$$x - g(x) = \lim_{N\to\infty} \sum_{\left\{\substack{C:k\leq N\\ l\leq N}\right\}} \frac{p}{N^2}\left[\rho'_N(C) + \rho''_N(C)\right]. \tag{A1.1}$$

Because $(1-x)^\alpha \leq 1$ for $\alpha \geq 0$ and $1/N^2 < 1/N$, we have

$$x - g(x) \leq \lim_{N\to\infty} \frac{15}{N} \sum_{\left\{\substack{C:k\leq N\\ l\leq N}\right\}} p x^p \tag{A1.2}$$

and then

$$x - g(x) \leq \lim_{N\to\infty} \frac{15}{N} \sum_{C} p\, x^p \tag{A1.3}$$

so that, as soon as the infinite series $\Sigma_C p x^p$ converges, we shall have $g(x) \equiv x$.

But we know that the percolation series converges (with sum $g(x)$). On the other hand, $q \leq 2p+2$ (the equality is obtained for linear clusters), so that:

$$\sum_{(C)} p x^p (1-x)^{2p+2} \leq g(x). \tag{A1.4}$$

Taking $x = \frac{1}{3}$, which maximises $x(1-x)^2$, one obtains:

$$\sum_{(C)} p\left(\frac{4}{27}\right)^p \leq \frac{9}{4} g(x). \tag{A1.5}$$

Thus, $g(x) \equiv x$ for $x \leq 4/27$.

Appendix 2.

Let us rewrite (30)–(33) of the text as:

$$(2^N-1)x=\sum_{\{C:k\leqslant N-1\}} yu(\alpha)+\sum_{\{C:k\leqslant N-1\}} y[v+\alpha w]+\sum_{\{C:k=N\}} yz \tag{A2.1}$$

with

$$\begin{aligned} y &= px^p(1-x)^{p+2}2^t \\ u &= 2^{N-k}-2+(2^{N-k}-\alpha)(1-x)^{-2n} \\ v &= (1-x)^{-1} \\ w &= (1-x)^{-2n} \\ z &= (1-x)^{-1-2n} \end{aligned} \tag{A2.2}$$

with $0\leqslant\alpha\leqslant 2$ being arbitrary.

Then

$$\begin{aligned} y_N(\alpha) &= \frac{1}{2^N-1}\sum_{\{C:k\leqslant N-1\}} yu \\ &\geqslant \frac{1}{2^N-1}\sum_{\{C:k\leqslant k_0\}} yu \end{aligned} \tag{A2.3}$$

for fixed k_0, so that

$$x\geqslant \lim_{N\to\infty} y_N \geqslant y_0=\sum_{\{C:k\leqslant k_0\}} px^p(1-x)^{p+2}\frac{2^t}{2^k}[1+(1-x)^{-2n}]. \tag{A2.4}$$

Now, let $k_0\to\infty$: this shows that the percolation series (36) is the limit of an increasing and bounded sequence, so that it converges and is less than x.

But, if $\frac{1}{2}\leqslant\alpha\leqslant 2$, one will see easily that:

$$y_N(\alpha)\leqslant\sum_{\{C:k\leqslant N-1\}} px^p(1-x)^{p+2}\frac{2^t}{2^k}[1+(1-x)^{-2n}]. \tag{A2.5}$$

(A2.4) and (A2.5) prove that:

$$\lim_{N\to\infty} y_N(\alpha)=g(x), \tag{A2.6}$$

the sum of the percolation series, which is independent of α.

Choosing now $\frac{1}{2}\leqslant\alpha_1\leqslant\alpha_2\leqslant 2$, one obtains:

$$0=\lim_{N\to\infty}\frac{1}{2^N-1}(\alpha_2-\alpha_1)\sum_{\{C:k\leqslant N-1\}} yw. \tag{A2.7}$$

As $v<w$, one also has

$$0=\lim_{N\to\infty}\frac{1}{2^N-1}\sum_{\{C:k\leqslant N-1\}} yv \tag{A2.8}$$

so that

$$x-g(x)=\lim_{N\to\infty}\frac{1}{2^N-1}\sum_{\{C:k=N\}} yz. \tag{A2.9}$$

If $x \neq 1$, this is less than

$$(1-x)^{-1} \frac{2^{N+1}-1}{2^N-1} \frac{1}{2^{N+1}-1} \sum_{\{C:k\leqslant N\}} yw \tag{A2.10}$$

which goes to zero.

Thus we have

$$g(x) \equiv x \qquad \text{for} \qquad x \neq 1 (0 \leqslant x < 1). \tag{A2.11}$$

Appendix 3.

We want to study

$$y = \lim_{N\to\infty} \frac{1}{N^2} \sum_{Cl\in C_r} \langle S_i \ldots S_n \rangle_N \tag{A3.1}$$

(see equation (77) of the text).

To this aim, we follow the arguments of Griffiths concerning equivalence between 'thermodynamic magnetisation and local magnetisation' (Griffiths 1967, p 484).

For fixed $(i, \ldots, n)$ we have:

$$\langle S_i \ldots S_n \rangle_N \leqslant \lim_{N\to\infty} \langle S_i \ldots S_n \rangle_N = \langle S_i \ldots S_n \rangle \tag{A3.2}$$

but this limit is the same for the $(N-1-l)(N-1-k)$ terms in (A3.1) because all these terms are corresponding terms in clusters which belong to the same type (translation invariance).

Thus

$$y \leqslant \langle S_i \ldots S_n \rangle \lim_{N\to\infty} \frac{(N-1-l)(N-1-k)}{N^2} = \langle S_i \ldots S_n \rangle. \tag{A3.3}$$

Now, let M be fixed, larger than l and k. For N sufficiently large, consider the square $(N-2M)\times(N-2M)$ centred on 0, and let $A_1, \ldots, A_t$ with $t = (N-2M-1-l)(N-2M-1-k)$ be the set $(i, \ldots, n)$ corresponding to clusters inside this square.

One has:

$$y \geqslant \lim_{N\to\infty} \frac{1}{N^2} [\langle \sigma_{A_1} \rangle_N + \ldots + \langle \sigma_{A_t} \rangle_N] \tag{A3.4}$$

where

$$\langle \sigma_A \rangle_N \text{ means } \left\langle \prod_{i\in A} S_i \right\rangle_N.$$

But any cluster inside the square $(N-2M)\times(N-2M)$ is also inside a square, centred on the left upper corner of the rectangle $(k \times l)$ containing the cluster, with size $M \times M$. All these squares, in turn, are inside the initial square $N \times N$.

Thus

$$\langle \sigma_{A_j} \rangle_N \geqslant \widetilde{\langle \sigma_{A_j} \rangle_M} \tag{A3.5}$$

where the right-hand side of (A3.5) is an average with Hamiltonian restricted to the squares $M \times M$.

But clearly

$$\widetilde{\langle \sigma_{A_j} \rangle}_M = \widetilde{\langle \sigma_{A_1} \rangle}_M \tag{A3.6}$$

so that

$$y_A \geqslant \frac{(N-2M-1-l)(N-2M-1-k)}{N^2} \widetilde{\langle \sigma_{A_k} \rangle}_M. \tag{A3.7}$$

For fixed M and for $N \to \infty$, one obtains:

$$y_A \geqslant \widetilde{\langle \sigma_{A_j} \rangle}_M. \tag{A3.8}$$

Now, if $M \to \infty$,

$$\widetilde{\langle \sigma_{A_j} \rangle}_M \to \langle \sigma_{A_j} \rangle. \tag{A3.9}$$

Finally, we thus have:

$$y = \langle \sigma_{A_j} \rangle \tag{A3.10}$$

which corresponds to equation (77) of the text.

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